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MASSIVE INTEGRABLE SOLITON THEORIES

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ABSTRACT

Massive integrable field theories in 1+1 dimensions are defined at the Lagrangian level, whose classical equations of motion are related to the “non-abelian” Toda field equations. They can be thought of as generalizations of the sine-Gordon and complex sine-Gordon theories. The fields of the theories take values in a non-abelian Lie group and it is argued that the coupling constant is quantized, unlike the situation in the sine-Gordon theory, which is a special case since its field takes values in an abelian group. It is further shown that these theories correspond to perturbations of certain coset conformal field theories. The solitons in the theories will, in general, carry non-abelian charges.

1. Introduction

In this letter we construct new series of integrable field theories in 1+1 dimensions. We show that the simplest examples are the sine-Gordon and complex sine-Gordon theories, respectively. The former is well-known to be integrable at the quantum level and an exact factorizable S-matrix has been found for the scattering of the classical lump solutions; in this case topological solitons and breathers [1]. Remarkably, the fundamental field of the theory is the ground-state of the breather system, so that all states in the theory can be understood from a semi-classical analysis of the lump solutions. Recently, the complex sine-Gordon theory has been quantized semi-classically and on the basis of this an exact factorizable S-matrix was proposed to describe the scattering of the lumps of the theory [2] (following earlier work [3]); in this case charged solitons. Like the sine-Gordon theory the fundamental field of the theory corresponds to one of classical lump solutions. These two theories serve as paradigms of what can be expected for the more complex situations: we expect the lump-like classical solutions to admit a semi-classical quantization from which the exact factorizable S-matrix could be deduced. In this sense the paper should be regarded as the first stage in a programme to find these exact S-matrices. The theories in general have a non-abelian global symmetry (the complex sine-Gordon is simpler since its symmetry group is abelian); so the solitons will carry non-abelian charges (as well as topological charges in some cases). In this sense they are similar to the Skyrme model in four dimensions (see for example [4]).

The classical integrable equations that underlie the theories are the “non-abelian Toda equations” of Leznov and Saveliev [5]. What was not so clear from their original work—and is the subject of the present paper—is whether these theories can be written in Lagrangian form and hence can be used to define relativistic quantum field theories. It transpires that many of these classical equations cannot be derived from a Lagrangian with a positive-definite kinetic term and a real potential term; however, there are several families for which this can be done. Most of the theories which admit a Lagrangian formalism, having a positive-definite kinetic energy and real potential energy, have massless degrees-of-freedom because the potential energy has flat directions. Whilst these theories may be interesting in their own right, they will not admit a factorizable S-matrix (at least in the conventional sense); however, we show how the massless degrees-of-freedom may be removed to give a purely massive integrable field theory.

In contrast with the usual Toda field theories, where the field takes values in the Cartan subgroup of a Lie group, the field $h(x, t)$ of the theories we shall discuss, takes values in a non-abelian Lie group G_0 . The kinetic term of the theory is simply the WZW

action for the group G_0 , so that the actions of the theories are of the form

$$S[h] = \frac{1}{\beta^2} \left\{ S_{\text{WZW}}[h] - \int d^2x V(h) \right\}, \quad (1.1)$$

where $V(h)$ is some potential function on the group manifold, to be specified, and β is a coupling constant which plays no role in the classical theory.¹ The potential has a minimum which can be chosen to be at the identity $h = 1$, and so expanding around the minimum by taking $h = 1 + i\phi + \dots$, one can see that the quadratic term in the kinetic energy is

$$\frac{1}{8\pi\beta^2} \int d^2x \text{Tr} (\partial_\mu \phi \partial^\mu \phi), \quad (1.2)$$

where $\text{Tr}(\)$ is the suitably normalized trace in some faithful representation of the Lie algebra \mathfrak{g}_0 associated to G_0 . If the kinetic term is to be positive-definite then immediately we see that the group G_0 must be the compact group, unless G_0 is abelian, in which case it can be both the compact or maximally non-compact group (in the latter case with $\beta^2 \rightarrow -\beta^2$ so that the kinetic term is positive definite rather than negative definite).

An important consequence of the form of the Lagrangian is that if the group G_0 is non-abelian (and compact) and the quantum theory is to be well-defined then the coupling constant has to be quantized:

$$\beta^2 = \frac{1}{k}, \quad k \in \mathbf{N}. \quad (1.3)$$

Such a quantization of the coupling constant does not occur in the sine-Gordon theory or the Toda theories because the group is abelian in these cases. So an important consequence of this is that in the quantum theory there will not be a continuous coupling constant; this will have important implications for the construction of the exact S-matrix of the theories. An example of this quantization of the coupling constant occurs in the complex sine-Gordon theory [2] which, as we have mentioned, is the simplest theory of this type with a non-abelian field.

2. The Leznov and Saveliev construction

We now turn to the definition of the potential $V(h)$. We will choose the potential so that the classical equations of motion are one of the integrable equations constructed by Leznov and Saveliev [5] (the so-called “non-abelian Toda equations”) and hence can be written in zero-curvature form.

¹ So in these theories, just as in the sine-Gordon theory, the semi-classical limit is the same as the weak-coupling limit.

The Leznov and Saveliev construction starts by considering an $\mathfrak{sl}(2)$ embedding of some finite Lie algebra g specified by the generators $\{J_{\pm}, J_0\}$. The Cartan element J_0 induces a gradation of g , by adjoint action:

$$g = \bigoplus_{k=-N}^N g_k, \quad [J_0, a] = ka \quad \text{for } a \in g_k. \quad (2.1)$$

In general, k runs over the half-integers because there are both integer and half-integer spin representations when g is decomposed under the $\mathfrak{sl}(2)$; however, the Leznov and Saveliev construction in its original form is restricted to integral embeddings, i.e. there can be no half-integer spin representations and k runs over the integers only. (By definition $J_{\pm} \in g_{\pm 1}$.)

The field $h(x, t)$ takes values in the group G_0 associated to the Lie algebra of the zero-graded component g_0 . The associated integrable equation is of the form

$$\partial_- (h^{-1} \partial_+ h) = m^2 [J_+, h^{-1} J_- h], \quad (2.2)$$

where $\partial_{\pm} = \partial_t \pm \partial_x$ are the light-cone derivatives and m is a mass scale. This equation can be written in zero-curvature form as

$$[\partial_+ + h^{-1} \partial_+ h + imJ_+, \partial_- - imh^{-1} J_- h] = 0. \quad (2.3)$$

In spite of the explicit mass scale, these equations are classically conformally invariant and are actually generalizations of the Liouville theory, which is recovered by taking $g = \mathfrak{sl}(2)$ with

$$h = \begin{pmatrix} e^{\phi} & 0 \\ o & e^{-\phi} \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.4)$$

Such conformally invariant theories with non-compact groups—hence indefinite kinetic terms—have been studied in the context of two-dimensional black-holes [6].

In this paper we wish to define massive field theories. Leznov and Saveliev showed how to generalize the aforementioned construction to break conformal invariance whilst preserving integrability (these theories have also been discussed in [7]). The idea involves generalizing equations (2.2) to

$$\partial_- (h^{-1} \partial_+ h) = m^2 [\Lambda_+, h^{-1} \Lambda_- h], \quad (2.5)$$

where $\Lambda_{\pm} = J_{\pm} + Y_{\pm}$. Integrability is then maintained if the constant elements $Y_{\pm} \in g_{\mp N}$, the minimal and maximal graded components of g , respectively. The conformal invariance is manifestly broken because the elements Λ_{\pm} do not have definite grade.

The Leznov and Saveliev construction of integrable equations describing massive fields has a lot freedom, firstly due to the choice of the $\mathfrak{sl}(2)$ embedding and secondly due to

choice of the elements Y_{\pm} . However, if we wish to describe theories that can be written at the Lagrangian level with a positive definite kinetic energy then we shall find that there are many fewer possibilities. First of all, we must choose the form of the group G_0 . As we have already pointed out, if g_0 is non-abelian, in order to get a theory with a positive-definite kinetic term we are forced to take the compact form of the group G_0 . This imposes a reality condition on the field $h^{\dagger} = h^{-1}$ which is consistent with the equations of motion (2.5) only if $\Lambda_{\pm}^{\dagger} = \Lambda_{\pm}$ and so $J_{\pm}^{\dagger} = Y_{\pm}$. But since $J_{\pm}^{\dagger} \in g_{\mp 1}$ this implies that the $\mathfrak{sl}(2)$ embedding has to have $N = 1$. However, in general, the resulting theory will not be invariant under parity. Such theories may be interesting but here we will study theories invariant under parity, for which we need $\Lambda_+ = \Lambda_-$.² This requires $Y_{\pm} = J_{\mp}$ and so $J_{\pm}^{\dagger} = J_{\mp}$; hence

$$\Lambda \equiv \Lambda_+ = \Lambda_- = J_+ + J_- . \quad (2.6)$$

The element Λ is actually an element of a Heisenberg subalgebra of the loop algebra of g . In fact the language of Heisenberg subalgebras, which is more suited to describing the associated integrable hierarchies of equations [8], can be used instead of the language of $\mathfrak{sl}(2)$ embedding; however for our purposes we will find it sufficient to stick to the language of $\mathfrak{sl}(2)$ embeddings.

To summarize: if g_0 is non-abelian then we can only construct a field theory with a positive-definite kinetic term if the $\mathfrak{sl}(2)$ embedding of g induces a gradation of the form

$$g = g_{-1} \oplus g_0 \oplus g_1 . \quad (2.7)$$

Hence, under the $\mathfrak{sl}(2)$, g decomposes into triplets and singlets only.

With these restrictions the equations of motion may be derived by extremizing the action

$$S[h] = \frac{1}{\beta^2} \left\{ S_{\text{WZW}}[h] - \frac{m^2}{8\pi} \int d^2x \text{Tr} (h \Lambda h^{-1} \Lambda) \right\} . \quad (2.8)$$

So the Lagrangian has the advertised form (1.1) with a potential

$$V(h) = \frac{m^2}{8\pi} \text{Tr} (h \Lambda h^{-1} \Lambda) . \quad (2.9)$$

Notice that since $\Lambda^{\dagger} = \Lambda$ the potential is real when h takes values in the compact group. The action (2.8) can be obtained by Hamiltonian reduction of the two loop WZW model associated to the affine untwisted Kac-Moody algebra $g^{(1)}$ [9]; this generalizes the well known relation between the reduced WZW model and the non-abelian conformal Toda models [10].

² This is clear from the equations of motion which have the symmetry $x_{\pm} \mapsto x_{\mp}$ along with $h \mapsto h^{-1}$, only if $\Lambda_+ = \Lambda_-$.

If g_0 is abelian then the above restriction to the compact group does not apply because G_0 can then be chosen to be either the compact or maximally non-compact group, corresponding to $h^\dagger = h^{-1}$ and $h^\dagger = h$, respectively. The abelian case arises when the $\mathfrak{sl}(2)$ embedding is the principal embedding in g . In this case

$$J_+ = \sum_{j=1}^r E_{\alpha_j}, \quad (2.10)$$

a sum over the step generators corresponding to the simple roots of g , and g_{-N} is spanned by the step generator E_{α_0} , corresponding to the lowest root. Therefore,

$$\Lambda_+ = \sum_{j=0}^r E_{\alpha_j}. \quad (2.11)$$

So with G_0 being the compact group we see that the equations of motion are not consistent with the reality condition except when $g = \mathfrak{su}(2)$ when the resulting theory is the sine-Gordon theory. On the other hand, if we choose the abelian group G_0 to be the maximally non-compact group then the equations of motion (2.5) are real by virtue of the fact that $h^\dagger = h$ and $\Lambda_+^\dagger = \Lambda_-$. These theories are the affine Toda field equations associated to an algebra g .

Returning to the non-abelian theories, we see that the potential has a minimum when $h = 1$, the identity in G_0 . If we expand the action around the minimum $h = \exp i\phi$, with $\phi^\dagger = \phi$, for small ϕ , we have

$$S = \frac{1}{8\pi\beta^2} \int d^2x \left\{ \text{Tr}(\partial_\mu \phi \partial^\mu \phi) - \frac{m^2}{2} \text{Tr}([\phi, \Lambda][\phi, \Lambda]) + \dots \right\}, \quad (2.12)$$

up to a constant.³ So we expect the quantum theory to contain a set of particles with masses found by diagonalizing these quadratic terms. It is straightforward to see that there will be massless particles associated to the subalgebra $g_0^0 \subset g_0$ of elements which commute with Λ . So the potential, therefore, will have flat directions and the theory will have a mixture of massless and massive degrees-of-freedom. Such theories may be interesting to study in there own right, since they describe renormalization group trajectories which interpolate between the WZW models based on G_0 and G_0^0 (the compact group associated to g_0^0).

However, in this paper we wish to study theories with a mass gap, because such theories will admit an S-matrix description. To achieve this, we have to somehow introduce the constraints that $P(h^{-1}\partial_+ h) = 0$ and $P(h\partial_- h^{-1}) = 0$, where P is the projection operator onto the subalgebra $g_0^0 \subset g_0$. The way to introduce this constraint was discussed

³ The WZ term is a total derivative at the quadratic order and hence does not contribute at this order.

in [11]. First of all, notice that the potential is invariant under the diagonal group action $h \mapsto \alpha h \alpha^{-1}$, for $\alpha \in G_0^0$. So the idea is to *gauge* this diagonal G_0^0 group action by introducing a gauge field A_\pm taking values in \mathfrak{g}_0^0 . The kinetic term is then the gauged WZW action $S_{\text{WZW}}[h, A_\pm]$ which describes the G_0/G_0^0 coset conformal field theory [12]. The action of the theory is then

$$S[h, A_\pm] = \frac{1}{\beta^2} \left\{ S_{\text{WZW}}[h, A_\pm] - \frac{m^2}{8\pi} \int d^2x \text{Tr} (h \Lambda h^{-1} \Lambda) \right\}. \quad (2.13)$$

The action of the gauged WZW model is explicitly [12]

$$\begin{aligned} S_{\text{WZW}}[h, A_\pm] &= S_{\text{WZW}}[h] \\ &+ \frac{1}{2\pi} \int d^2x \text{Tr} (A_+ h \partial_- h^{-1} + A_- h^{-1} \partial_+ h + A_+ h A_- h^{-1} - A_+ A_-). \end{aligned} \quad (2.14)$$

The variation of (2.13) with respect to h gives the equation of motion which can be written in the zero-curvature form

$$[\partial_+ + h^{-1} \partial_+ h + h^{-1} A_+ h + i m z \Lambda, \partial_- + A_- - i m z^{-1} h^{-1} \Lambda h] = 0, \quad (2.15)$$

where we have introduced the spectral parameter z whose inclusion plays an important role in establishing the integrability of the equations. Variations with respect to the gauge field leads to the constraints

$$\begin{aligned} P(h \partial_- h^{-1} + h A_- h^{-1}) - A_- &= 0, \\ P(h^{-1} \partial_+ h + h^{-1} A_+ h) - A_+ &= 0, \end{aligned} \quad (2.16)$$

where P is, as before, the projector onto the subalgebra \mathfrak{g}_0^0 . By projecting (2.15) onto \mathfrak{g}_0 one can see that the gauge field is flat: $[\partial_+ + A_+, \partial_- + A_-] = 0$ which reflects the vector gauge invariance of the action.

To show that the constraints (2.16) remove the massless degrees-of-freedom it suffices to choose the gauge $A_+ = A_- = 0$, which is consist due to the flatness of the gauge field and the vector gauge invariance of the action. In this gauge the equations of motion (2.15) reduce to

$$\partial_- (h^{-1} \partial_+ h) = m^2 [\Lambda, h^{-1} \Lambda h], \quad (2.17)$$

along with the constraints (2.16):

$$P(h^{-1} \partial_+ h) = 0, \quad P(h \partial_- h^{-1}) = 0. \quad (2.18)$$

As pointed out in [11], the constraints cannot be solved locally in this gauge; however, there are different gauge choices for which the constraint equations can be solved in terms of local fields.

So the theories that we end up with can be considered as integrable perturbations of certain coset conformal field theories. Perturbations of these conformal field theories have been considered recently in [11,13]. In the latter reference, it was shown how a perturbation of a G_0/G_0^0 coset model of the form $\text{Tr}(hTh^{-1}\bar{T})$ was classically integrable if the constant elements $T, \bar{T} \in g_0$ lie in the centralizer of g_0^0 . Notice that this construction is rather similar to the one discussed here, except that in our models Λ , the analogue of T , lies outside g_0 in the larger algebra g . However, in section 4 we shall find some overlap between these two formalisms.

Notice that the theories have a global G_0^0 symmetry. This is because the potential in (2.13), and therefore the action itself, is invariant under both left and right action by G_0^0 : $h \rightarrow \alpha h$ and $h \rightarrow h\alpha$, $\alpha \in G_0^0$. Therefore after gauging the diagonal subgroup there is a residual G_0^0 symmetry.

The conformal dimension of the perturbing operator can be found by taking an operator product expansion with the stress tensor of the coset model. This highlights an important property of these perturbations: in general the dimension of the perturbing operator depends upon the representation chosen for g for defining the potential. So the formalism potentially generates a number of inequivalent integrable perturbations of the coset model.

3. $\mathfrak{sl}(2)$ embeddings with $N = 1$ and the models

In the appendix we explain how to derive all the $\mathfrak{sl}(2)$ embeddings with $N = 1$. In this section we put these results together with the formalism of the last section to write down integrable perturbations of various coset models.

Notice from the form of the action (2.8), and the invariance of the Haar measure in the path integral, that theories related by the fact that Λ_1 and Λ_2 are conjugate, so that $\Lambda_1 = \alpha\Lambda_2\alpha^{-1}$ for a constant element $\alpha \in G_0$, are actually identical since they are related by a transformation of the field. This means that we need only consider $\mathfrak{sl}(2)$ embeddings up to conjugation by the group G_0 . In addition, we are restricting ourselves to the parity invariant theories, so that $J_\pm^\dagger = J_\mp$.

For the classical Lie algebras we shall use the orthonormal vectors $\{e_j\}$ and the dual basis $\{e_j^\vee\}$, with $e_j^\vee(e_k) = \delta_{jk}$, in terms of which one can describe both the root space and the Cartan subalgebra of g , respectively.

For A_1 the principal embedding itself has $N = 1$. In the two-dimensional representation

$$J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.1)$$

The subalgebra g_0 consists of the Cartan subalgebra and $g_0^0 = \emptyset$. In this case the field is valued in $U(1)$ and in the next section we show that it is the sine-Gordon theory.

For general A_{n-1} , n has to be even, $n = 2p$, and there is an $N = 1$ embedding, up to conjugation, for which

$$J_0 = \frac{1}{2} \sum_{j=1}^p (e_j^\vee - e_{p+j}^\vee), \quad J_+ = \sum_{j=1}^p E_{e_j - e_{p+j}}. \quad (3.2)$$

The compact form of the algebra is $\mathfrak{su}(2p)$ where an element of the algebra is a traceless hermitian matrix.⁴ In the defining $2p$ -dimensional representation we may take

$$J_0 = \frac{1}{2} \begin{pmatrix} 1_p & 0 \\ 0 & -1_p \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}. \quad (3.3)$$

So the subalgebra g_0 consists of elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a^\dagger = a, \quad b^\dagger = b, \quad \text{Tr}(a) + \text{Tr}(b) = 0; \quad (3.4)$$

hence $g_0 = \mathfrak{su}(p) \oplus \mathfrak{su}(p) \oplus \mathfrak{u}(1)$. The subalgebra g_0^0 consists of elements which commute with Λ and thus of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a^\dagger = a, \quad \text{Tr}(a) = 0; \quad (3.5)$$

hence $g_0^0 = \mathfrak{su}(p)$: the diagonal embedding. So the model corresponds to an integrable perturbation of the coset

$$\frac{\text{SU}(p) \times \text{SU}(p)}{\text{SU}(p)} \times U(1). \quad (3.6)$$

An element of the group G_0 can be written as

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad h_1, h_2 \in U(p), \quad \det(h_1)\det(h_2) = 1. \quad (3.7)$$

For this particular representation the potential has the form

$$V(h_1, h_2) = \frac{m^2}{8\pi} \text{Tr} (h_1 h_2^{-1} + h_2 h_1^{-1}). \quad (3.8)$$

The case when $p = 2$ is discussed more fully in the next section.

For C_n there is only one embedding with $N = 1$, up to conjugation; namely

$$J_0 = \frac{1}{2} \sum_{j=1}^n e_j^\vee, \quad J_+ = \sum_{j=1}^n E_{2e_j}, \quad (3.9)$$

⁴ In our convention the generators of the compact Lie algebra will be hermitian.

which follows from (A.9) with $p = 0$ and $r = n$. The compact form of the algebra is $\mathfrak{sp}(n)$ ⁵ where an element of the algebra in the defining $2n$ -dimensional representation has the block form

$$\begin{pmatrix} a & b \\ b^\star & -a^\star \end{pmatrix}, \quad a^\dagger = a, \quad b^T = b, \quad (3.10)$$

where a and b are n dimensional matrices. In this representation we can take

$$J_0 = \frac{1}{2} \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}. \quad (3.11)$$

The subalgebra g_0 consist of elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & -a^\star \end{pmatrix}, \quad a^\dagger = a; \quad (3.12)$$

hence $g_0 = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. The elements which commute with Λ are of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a^\star = -a, \quad a^T = -a, \quad (3.13)$$

and so $g_0^0 = \mathfrak{so}(n)$. So the model corresponds to a perturbation of the coset

$$\frac{\mathbf{U}(n)}{\mathbf{SO}(n)}. \quad (3.14)$$

In this representation, an element of G_0 has the form

$$h = \begin{pmatrix} \tilde{h} & 0 \\ 0 & \tilde{h}^\star \end{pmatrix}, \quad \tilde{h} \in \mathbf{U}(n), \quad (3.15)$$

and the potential is

$$V(\tilde{h}) = \frac{m^2}{8\pi} \text{Tr} \left(\tilde{h} \tilde{h}^T + \tilde{h}^\star \tilde{h}^{-1} \right). \quad (3.16)$$

We now turn to the algebras of the orthogonal groups. For both B_r and D_r there is an $N = 1$ embedding with

$$J_0 = e_1^\vee, \quad J_+ = E_{e_1 - e_2} + E_{e_1 + e_2}, \quad (3.17)$$

where we have realized the embedding in terms of a regular subalgebra $D_2 \subset B_r$ or D_r . The compact form of the algebras is $\mathfrak{so}(n)$ (corresponding to B_r if $n = 2r + 1$ and D_r

⁵ In our notation $\mathfrak{sp}(n)$ has rank n .

if $n = 2r$). The defining representation consists of purely imaginary antisymmetric n -dimensional matrices. In this representation we can take

$$J_0 = \frac{i}{2} \begin{pmatrix} 0 & 1 & \cdots \\ -1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \Lambda = i \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.18)$$

where the elements not shown are zero. The subalgebra g_0 consists of matrices of the form

$$i \begin{pmatrix} 0 & a & \cdots \\ -a & 0 & \cdots \\ \vdots & \vdots & b \end{pmatrix}, \quad (3.19)$$

where a is a real number and b is a real $(n-2)$ -dimensional antisymmetric matrix; hence $g_0 = \mathfrak{so}(n-2) \oplus \mathfrak{u}(1)$. The elements which commute with Λ are those of the form

$$i \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & c \end{pmatrix}, \quad (3.20)$$

where c is a real $(n-3)$ -dimensional antisymmetric matrix; therefore $g_0^0 = \mathfrak{so}(n-3)$. So the model corresponds to an integrable perturbation of the coset

$$\frac{\mathrm{SO}(n-2)}{\mathrm{SO}(n-3)} \times \mathrm{U}(1). \quad (3.21)$$

In this representation the field has the block form

$$h = \begin{pmatrix} A & 0 \\ 0 & \tilde{h} \end{pmatrix}, \quad \tilde{h} \in \mathrm{SO}(n-2), \quad (3.22)$$

where the $\mathrm{U}(1)$ factor is

$$A = \exp \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \quad (3.23)$$

In this case the potential cannot be written down in a neat way. The models admit a reduction, preserving integrability, which involves restricting the field to be an element of the subgroup $\mathrm{SO}(n-2) \subset \mathrm{SO}(n-2) \times \mathrm{U}(1)$. To see that this is consistent with the equations of motion, one only has to notice that $[\Lambda, h^{-1} \Lambda h] \in \mathfrak{so}(n-2)$ when $A = 1$. Hence, the reduced theory will describe an integrable deformation of the coset $\mathrm{SO}(n-2)/\mathrm{SO}(n-3) \sim \mathrm{S}^{n-3}$. The example with $n = 5$ is considered more explicitly in the next section.

For D_{2p} , where p is an integer, there is an additional $N = 1$ embedding specified by

$$J_0 = \frac{1}{2} \sum_{j=1}^{2p} e_j^\vee, \quad J_+ = \sum_{j=1}^p E_{e_{2j-1} + e_{2j}}. \quad (3.24)$$

The compact form of the algebra is $\mathfrak{so}(4p)$. In the $4p$ -dimensional defining representation, elements of the algebra correspond to purely imaginary antisymmetric matrices. In this representation we can take, in block form,

$$J_0 = \frac{i}{2} \begin{pmatrix} 0 & 1_{2p} \\ -1_{2p} & 0 \end{pmatrix}, \quad \Lambda = i \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad (3.25)$$

where the $2p$ -dimensional matrix j is in block form

$$j = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}. \quad (3.26)$$

Elements of the subalgebra g_0 have the following block form

$$i \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^T = -a, \quad b^T = b, \quad (3.27)$$

where both a and b are real $2p$ -dimensional matrices. We can amalgamate these to form $2p$ -dimensional hermitian matrix $b + ia$, showing that $g_0 = \mathfrak{su}(2p) \oplus \mathfrak{u}(1)$. The elements of g_0 which commute with Λ are those for which $[a, j] = 0$ and $\{b, j\} = 0$. These conditions can be written

$$j(b + ia) = -(b + ia)^T j, \quad (3.28)$$

which is the defining relation for the Lie algebra $\mathfrak{sp}(p)$. Hence the model corresponds to a perturbation of the coset

$$\frac{\mathbf{U}(2p)}{\mathbf{Sp}(p)}. \quad (3.29)$$

For D_{2p} there is an additional non-conjugate $\mathfrak{sl}(2)$ embedding got by replacing $e_{2p} \mapsto -e_{2p}$ in (3.24). However, although the embedding is non-conjugate it is related by a diagram symmetry which means that the resulting theory is equivalent.

For the exceptional Lie algebras, there is only one $\mathfrak{sl}(2)$ embedding with $N = 1$, which is in E_7 . In this case the model will correspond to some integrable perturbation of a coset of $E_6 \times \mathbf{U}(1)$, although it is not very instructive to write down the potential in this case.

Notice that for all these embeddings g_0 has a $\mathbf{U}(1)$ factor. This factor can be traced to the generator $J_0 \in g_0$ which commutes with the rest of g_0 by construction and hence always forms a $\mathfrak{u}(1)$ subalgebra. As we have seen and discuss more fully in the next section, for some of the models this $\mathbf{U}(1)$ field can be decoupled whilst preserving integrability.

4. Some explicit examples

In this section we will consider more explicitly some of the theories defined above.

The first example we consider is the principal embedding in A_1 . In the two-dimensional representation of (3.1) the field is

$$h = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad (4.1)$$

and the action can be written explicitly in terms of the field ϕ as

$$S[\phi] = \frac{1}{8\pi\beta^2} \int d^2x \left((\partial_\mu \phi)^2 - m^2 \cos 2\phi \right). \quad (4.2)$$

This example is nothing but the ubiquitous sine-Gordon theory. In this case the WZ-term does not contribute because the group is abelian, and so correspondingly there will be no quantization of the coupling constant in the quantum theory. The spectrum of the model consists of solitons and breathers whose exact S-matrix was written down in [1]. Notice that all the states of the theory appear as the quantization of classical lump solutions.

The second example we consider, and in some respects the simplest after the sine-Gordon theory, since g_0 has the smallest rank, is the theory based on B_2 or C_2 . In fact it is easier to work with C_2 since its vector representation has dimension 4 (corresponding to the spinor representation of B_2). There is one embedding, up to conjugation, which in C_2 we can take to be

$$J_+ = E_{e_1+e_2}, \quad (4.3)$$

this being conjugate to (3.9). So in 2×2 block form

$$\Lambda = J_+ + J_- = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \quad (4.4)$$

where σ_1 is one of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.5)$$

In the same block form, the field $h \in G_0$ is from (3.15)

$$h = \begin{pmatrix} \tilde{h} & 0 \\ 0 & \tilde{h}^\star \end{pmatrix}, \quad (4.6)$$

where \tilde{h} is an element of the group $U(2)$ ($\simeq SO(3) \times U(1)$) in the two-dimensional representation. The group G_0^0 consists of elements in the Cartan subalgebra of the $SU(2)$ factor; hence, elements of the form

$$\begin{pmatrix} e^{ia} & 0 & 0 & 0 \\ 0 & e^{-ia} & 0 & 0 \\ 0 & 0 & e^{-ia} & 0 \\ 0 & 0 & 0 & e^{ia} \end{pmatrix}, \quad (4.7)$$

for arbitrary a . The equations of motion (2.15) decouple into two equations involving the 2×2 matrix \tilde{h} , which are complex conjugate to each other. Before gauge fixing one of the equations is

$$\left[\partial_+ + \tilde{h}^{-1} \partial_+ \tilde{h} + \tilde{h}^{-1} A_+ \tilde{h}, \partial_- + A_- \right] + m^2 \left(\sigma_1 \tilde{h}^T \sigma_1 \tilde{h} - \tilde{h}^{-1} \sigma_1 \tilde{h}^* \sigma_1 \right) = 0, \quad (4.8)$$

where the gauge fields A_{\pm} take values in the Cartan subalgebra of the $SU(2)$, i.e. its components are proportional to σ_3 .

As we mentioned in the last section, the equations of motion admit a reduction—preserving integrability—by restricting $\tilde{h} \in SU(2)$ so that the reduced theory is actually a perturbation of the coset $SU(2)/U(1)$. In this case we will write the equations of motion in terms of local fields which means that we must choose a gauge different from $A_{\pm} = 0$. One way to fix the gauge is to choose a parameterization of the form (as in [11])

$$\tilde{h} = \begin{pmatrix} u & i\sqrt{1-uu^*} \\ i\sqrt{1-uu^*} & u^* \end{pmatrix}. \quad (4.9)$$

The constraints (2.16) can then be solved for the gauge field:

$$A_+ = \frac{u^* \partial_+ u - u \partial_+ u^*}{4(1-uu^*)} \sigma_3, \quad A_- = \frac{u \partial_- u^* - u^* \partial_- u}{4(1-uu^*)} \sigma_3. \quad (4.10)$$

The equations of motion in this gauge are then from (4.8)

$$\partial_- \partial_+ u + \frac{u^* \partial_+ u \partial_- u}{1-uu^*} + m^2 u(1-uu^*) = 0, \quad (4.11)$$

and its complex conjugate. This is precisely the complex sine-Gordon equation. The fact that this theory could be derived from perturbing a coset model was first pointed out in [13] and discussed further in [11]. What we have found is that the theory fits quite naturally into the class of non-abelian Toda theories, a fact which was not apparent in [11].

The theory exhibits charged soliton solutions [14,3], where the charge corresponds to the residual $U(1)$ symmetry $u \rightarrow e^{i\theta} u$. Furthermore it has been argued that the theory is integrable at the quantum level [2,3]. The theory corresponds to an integrable perturbation

of the $SU(2)/U(1)$ coset model and in this case the dimension of the perturbing operator is $\Delta = \bar{\Delta} = 2/(k+2)$, independent of the representation chosen for the potential. The solitons have internal motion which is indicative of the existence of the global $U(1)$ symmetry and rather surprisingly the particle states of the theory are actually identified with particular solitons (in much the same way that the particle of the sine-Gordon theory corresponds to a breathing solution). The full S-matrix of the theory has been found in [2].

The final set of models we consider are those associated to $su(2p)$. Let us take the vector representation in which Λ is given in (3.3). An element of the group $G_0 = SU(p) \times SU(p) \times U(1)$ is of the form

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad (4.12)$$

where $h_1, h_2 \in U(p)$, with $\det(h_1)\det(h_2) = 1$. The equations of motion in the $A_{\pm} = 0$ gauge are

$$\partial_-(h_1^{-1}\partial_+h_1) = m^2(h_2^{-1}h_1 - h_1^{-1}h_2), \quad h_1^{-1}\partial_+h_1 + h_2^{-1}\partial_+h_2 = 0. \quad (4.13)$$

When $p = 2$, as for the theory associated to $so(5)$, the theory admits an integrable reduction by setting $\det(h_1) = \det(h_2) = 1$, which is clearly consistent with (4.13). In this case the theory describes an integrable deformation of the coset model $SU(2) \times SU(2)/SU(2)$. This is precisely the model discussed in [15]. The conformal dimension of the perturbation in this case is $\Delta = \bar{\Delta} = 3/(4 + 2k)$. These equations have soliton solutions [16] and it would be interesting to proceed to a semi-classical quantization and hence determine the exact S-matrix. In these theories the perturbation does depend on the representation chosen for g_0 . So if we choose h_1 and h_2 to be in the representation of spin j then one can show that the perturbation of the coset model has dimension $\Delta = \bar{\Delta} = 2j(j+1)/(k+2)$.

5. Discussion

We have constructed $1 + 1$ dimensional field theories which are classically integrable. The classical equations of motion are related to the non-abelian Toda equations and these equations admit soliton solutions [7]. These theories are naturally to be thought of as generalizations of the sine-Gordon and complex sine-Gordon theories. So we expect that, in common with those theories, the spectrum of quantum states can be understood in terms of the semi-classical quantization of various lump-like solutions (solitons and breathers). The theories in general have a global non-abelian symmetry group and so we expect that the soliton solutions will transform in representations of the symmetry group. We pointed out an essential difference between the theories whose fields take values in a group whose

algebras are non-abelian and the sine-Gordon theory, where the group is abelian: in the former the coupling constant is quantized at the quantum level.

We have shown that these theories can be viewed as integrable perturbations of certain coset conformal field theories and using this picture should allow one to establish whether integrability survives at the quantum level, via the method of Zamolodchikov [17].

We have seen that in certain cases it was possible to perform a reduction of the models, whilst maintaining integrability. In fact there are a number of possibilities for making such reductions. In general, the idea is that if there exists a subalgebra $\tilde{g} \subset g_0$, with corresponding compact group $\tilde{G} \subset G_0$, and $[\Lambda, \tilde{h}^{-1} \Lambda \tilde{h}] \in \tilde{g}$ for $\tilde{h} \in \tilde{G}$, then the reduction of the model to \tilde{G} will be integrable. So as well as the reductions already mentioned, there are many other possibilities; two examples are

$$\frac{\mathrm{SU}(p) \times \mathrm{SU}(p)}{\mathrm{SU}(p)} \times \mathrm{U}(1) \rightarrow \frac{\mathrm{SO}(p) \times \mathrm{SO}(p)}{\mathrm{SO}(p)}, \quad (5.1)$$

and

$$\frac{\mathrm{SU}(2q) \times \mathrm{SU}(2q)}{\mathrm{SU}(2q)} \times \mathrm{U}(1) \rightarrow \frac{\mathrm{Sp}(q) \times \mathrm{Sp}(q)}{\mathrm{Sp}(q)}. \quad (5.2)$$

The outstanding problem is to find the classical spectrum of solitons and breathers of these theories and then perform a semi-classical quantization. From this it should be possible to infer the exact S-matrices.

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Appendix A:

The $\mathrm{sl}(2)$ embeddings $\{J_{\pm}, J_0\}$ of simple Lie algebras have been classified by Dynkin [18]; J_0 is called the *defining vector* and it characterizes the embedding up to conjugation by an automorphism of g . It is always possible to choose a system of simple roots of the algebra $\{\alpha_1, \dots, \alpha_r\}$ such that the numbers $s_i = \alpha_i(J_0)$ are 0, 1/2, or 1. These numbers are associated to the nodes of the Dynkin diagram of g , and the diagram, with the numbers written on, is called the *characteristic* of the embedding. A necessary and sufficient condition for two $\mathrm{sl}(2)$ subalgebras to be conjugate is that their characteristics coincide. Moreover, a necessary and sufficient condition that the embedding be

integral is that all the s_i 's are either 0 or 1. We will express the characteristic as a vector $\mathbf{s} = (s_1, \dots, s_r)$, and we shall use that particular choice of the system of simple roots of g from now on.

Let us restrict ourselves to integral embeddings and consider a positive root of g

$$\alpha = \sum_{i=1}^r n_i \alpha_i, \quad \text{with } n_i \in \mathbf{Z} \geq 0 \quad \text{for all } i = 1, \dots, r; \quad (\text{A.1})$$

then

$$[J_0, E_{\pm\alpha}] = \pm \left(\sum_{i=1}^r n_i s_i \right) E_{\pm\alpha}. \quad (\text{A.2})$$

Therefore, if

$$\theta = \sum_{i=1}^r k_i \alpha_i \quad (\text{A.3})$$

is the highest root of g , where $\{k_1, \dots, k_r\}$ are the Kac labels of g , then, in eq. (2.1), $E_{\pm\theta} \in g_{\pm N}$ with

$$N = \sum_{i=1}^r k_i s_i. \quad (\text{A.4})$$

Consequently, $N = 1$ if, and only if, the characteristic of the embedding is such that $s_i = \delta_{ij}$ with $k_j = 1$.

Moreover, let i_1, \dots, i_p be all the indices for which $s_{i_1} = \dots = s_{i_p} = 0$, then, the zero-graded subalgebra g_0 is isomorphic to a direct sum of the $(r-p)$ -dimensional centre and the semisimple Lie algebra whose Dynkin diagram is the subdiagram of the Dynkin diagram of g consisting of the nodes i_1, \dots, i_p . We can now list all the $\mathfrak{sl}(2)$ embeddings of a simple Lie algebra which correspond to gradations having $N = 1$.

For the exceptional Lie algebras, their Kac labels show that the only possibilities are either E_6 , with $\mathbf{s} = (1, 0, 0, 0, 0, 0)$ or $\mathbf{s} = (0, 0, 0, 0, 1, 0)$, or E_7 , with $\mathbf{s} = (0, 0, 0, 0, 0, 1, 0)$. The characteristics of the different $\mathfrak{sl}(2)$ embeddings for these Lie algebras can be found in [18]⁶, and the only embedding with the required characteristic corresponds to E_7 with $\mathbf{s} = (0, 0, 0, 0, 0, 1, 0)$, which is associated to a regular subalgebra $3A_1 \subset E_7$, and whose corresponding zero-graded subalgebra is

$$(E_7)_0 = E_6 \oplus \mathfrak{u}(1). \quad (\text{A.5})$$

Therefore, we conclude that this is the only $\mathfrak{sl}(2)$ embedding with $N = 1$ of the exceptional Lie algebras G_2 , F_4 , E_6 , E_7 , and E_8 .

⁶ In [18], the characteristic is normalized to be $2\mathbf{s}$ and its components equal 0, 1, or 2 (0 or 2 if the embedding is integral).

For the classical Lie algebras, the $\mathfrak{sl}(2)$ subalgebras can be realized as the principal $\mathfrak{sl}(2)$ subalgebra of a regular subalgebra of g , up to a few exceptions occurring in D_n —for a review, see, for example [19], where very detailed expressions for the defining vectors of the embeddings are provided.

For A_n , as all the Kac labels equal one, $s_i = \delta_{ij}$ for some $j = 1, \dots, n$; but, using the results of [19] it can be checked that there is an embedding with this characteristic only when n is odd, $n = 2p - 1$, and $j = p$. Then

$$\mathbf{s} = (\underbrace{0, \dots, 0}_{p-1 \text{ times}}, 1, \underbrace{0, \dots, 0}_{p-1 \text{ times}}), \quad (\text{A.6})$$

and it is associated to the regular subalgebra

$$\underbrace{A_1 \oplus \dots \oplus A_1}_{p \text{ times}} \subset A_{2p-1}. \quad (\text{A.7})$$

In this case, the zero-graded subalgebra is

$$(A_{2p-1})_0 = A_{p-1} \oplus A_{p-1} \oplus \mathfrak{u}(1). \quad (\text{A.8})$$

For C_n , the only possibility allowed by the Kac labels is $\mathbf{s} = (0, \dots, 0, 1)$, and there is only one embedding with this characteristic. It can be realized in terms of any of the following regular subalgebras

$$\underbrace{A_1 \oplus \dots \oplus A_1}_{p \text{ times}} \oplus \underbrace{C_1 \oplus \dots \oplus C_1}_{r \text{ times}} \subset C_n, \quad (\text{A.9})$$

where p and r are arbitrary non-negative integers such that $n = 2p + r$, and C_1 is a regular A_1 subalgebra of C_n corresponding to a long root; all these different realizations are conjugate by inner automorphisms of C_n . The zero-graded subalgebra is

$$(C_n)_0 = A_{n-1} \oplus \mathfrak{u}(1). \quad (\text{A.10})$$

For B_n , the value of the Kac labels leads to $\mathbf{s} = (1, 0, \dots, 0)$, and there is only one embedding with this characteristic. It can be realized in terms of any of the two regular subalgebras

$$B_1 \subset B_n \quad \text{and} \quad D_2 \subset B_n, \quad (\text{A.11})$$

where B_1 is a regular A_1 subalgebra associated to a short root, and D_2 is a regular $A_1 \oplus A_1$ subalgebra with $J_+ = E_{e_j - e_{j+1}} + E_{e_j + e_{j+1}}$ for some $j = 1, \dots, n-1$; these two realizations are also conjugate by an inner automorphism of B_n . The zero-graded subalgebra is

$$(B_n)_0 = B_{n-1} \oplus \mathfrak{u}(1). \quad (\text{A.12})$$

Finally, for D_n , the Kac labels admit three possibilities. The first one is $\mathbf{s} = (1, 0, \dots, 0)$, and there is an $\mathfrak{sl}(2)$ embedding with this characteristic associated to the regular subalgebra

$$D_2 \subset D_n . \quad (\text{A.13})$$

In this case, the zero-graded subalgebra is

$$(D_n)_0 = D_{n-1} \oplus \mathfrak{u}(1) . \quad (\text{A.14})$$

The other two possibilities arise when n is even, $n = 2p$; they are

$$\mathbf{s} = (0, \dots, 0, 1) \quad \text{and} \quad \mathbf{s}' = (0, \dots, 1, 0) , \quad (\text{A.15})$$

and there are two embeddings associated to two non-conjugate regular subalgebras of the form

$$\underbrace{A_1 \oplus \dots \oplus A_1}_{p \text{ times}} \subset D_{2p} , \quad (\text{A.16})$$

whose corresponding zero-graded subalgebras are

$$(D_{2p})_0 = A_{2p-1} \oplus \mathfrak{u}(1) . \quad (\text{A.17})$$

These last two $\mathfrak{sl}(2)$ embeddings are conjugate by the diagram automorphism of D_{2p} that takes $\alpha_{2p-1} \leftrightarrow \alpha_{2p}$, but not by any inner automorphism of D_{2p} . Finally, let us mention that for D_4 the three embeddings are conjugate by diagram automorphisms.

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